(27) QN. → If $u = \cos^{-1}\dfrac{x+y}{\sqrt{x}+\sqrt{y}}$, show that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + \frac{1}{2}\cot u = 0.$$

Ans. → $\because u = \cos^{-1}\dfrac{(x+y)}{\sqrt{x}+\sqrt{y}}$

$$\cos u = \frac{x+y}{\sqrt{x}+\sqrt{y}} = \frac{x\left(1+\frac{y}{x}\right)}{\sqrt{x}\left(1+\frac{\sqrt{y}}{\sqrt{x}}\right)} = x^{\frac{1}{2}} f\left(\sqrt{\frac{y}{x}}\right) = u$$

∴ Here u is a homogeneous function of x and y of degree $\frac{1}{2}$.

Hence, from Euler's theorem.

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}\times u = \frac{1}{2}\cos u \quad -(2)$$

$\because u = \cos u$

$$\frac{\partial u}{\partial x} = -\sin u\frac{\partial u}{\partial x}$$

or, $x\dfrac{\partial u}{\partial x} = -x\sin u\dfrac{\partial u}{\partial x} \quad -(a)$

similar $y\dfrac{\partial u}{\partial y} = -y\sin u\dfrac{\partial u}{\partial y} \quad -(b)$

(a) + (b)

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = -\sin u\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)$$

$$\frac{1}{2}\cos u = -\sin u\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)$$

or, $-\frac{1}{2}\cot u = x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$

$\therefore x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + \frac{1}{2}\cot u = 0$ proved.

(18) If $u = \sin^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}\tan u$.

Ans → $\because u = \sin^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$

$\therefore \sin u = \frac{x+y}{\sqrt{x}+\sqrt{y}} = \frac{x\left(1+\frac{y}{x}\right)}{\sqrt{x}\left(1+\frac{\sqrt{y}}{\sqrt{x}}\right)} = x^{\frac{1}{2}} f\left(\sqrt{\frac{y}{x}}\right) = v$

Here v is homogeneous function of x and y of degree $\frac{1}{2}$.

Hence, from Euler's theorem.

$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = \frac{1}{2} \times v$ .... (1)

Now, $v = \sin u$ — (2)

$\therefore$ Diff. (2), partially w.r.t. x, we have

$\frac{\partial v}{\partial x} = \cos u \frac{\partial u}{\partial x}$

$x\frac{\partial v}{\partial x} = x\cos u\frac{\partial u}{\partial x}$ — (a)

Similarly $y\frac{\partial v}{\partial y} = y\cos u\frac{\partial u}{\partial y}$ — (b)

(a) + (b)

$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = \cos u\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)$

$\frac{1}{2}v = \frac{1}{2}\sin u = \cos u\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)$

or, $\frac{1}{2}\tan u = x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$. proved.

(30) If $u = \sin^{-1}\frac{x^2+y^2}{x+y}$, Show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \tan u$

Ans $\rightarrow$ $\because u = \sin^{-1}\frac{x^2+y^2}{x+y}$

$$\therefore \sin u = \frac{x^2+y^2}{x+y} = \frac{x^2\left(1+\frac{y^2}{x^2}\right)}{x\left(1+\frac{y}{x}\right)} = x f\left(\frac{y}{x}\right) = v$$

Hence, v is a homogeneous function of x and y of degree 1 (one)

Hence, from Euler's theorem

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = 1 \times v$$

Now, $v = \sin u$ — (2)

$\therefore$ Diff. (2), partially w.r.t. x, we have,

$$\frac{\partial v}{\partial x} = \cos u \frac{\partial u}{\partial x}$$

$$x\frac{\partial v}{\partial x} = x \cos u \frac{\partial u}{\partial x} \quad \text{— (a)}$$

Similarly $y\frac{\partial v}{\partial y} = y\cos u\frac{\partial u}{\partial y}$ — (b)

(a) + (b)

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = \cos u\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)$$

$$v = \sin u = \cos u\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)$$

or, $\frac{\sin u}{\cos u} = x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$

or, $\tan u = x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$ proved.

(31) If $u = \tan^{-1}\frac{x^2+y^2}{x-y}$, Show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}\sin 2u$.

Ans. → $\because u = \tan^{-1}\frac{x^2+y^2}{x-y}$

$\therefore \tan u = \frac{x^2+y^2}{x-y} = \frac{x^{\cancel{2}}\left(1+\frac{y^2}{x^2}\right)}{\cancel{x}\left(1-\frac{y}{x}\right)} = \frac{x\left(1+\frac{y^2}{x^2}\right)}{\left(1-\frac{y}{x}\right)} = v$

when v = homogeneous function of x and y of degree one

Hence, from Euler's theorem

$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = 1 \times v$

Now, $v = \tan u$ — ①

$\therefore$ Diff. ①, partially w.r.t. x, we have,

$\frac{\partial v}{\partial x} = \sec^2 u\frac{\partial u}{\partial x}$ — (a)

$x\frac{\partial v}{\partial x} = x\sec^2 u\frac{\partial u}{\partial x}$ — (a)

Similarly $y\frac{\partial v}{\partial y} = y\sec^2 u\frac{\partial u}{\partial y}$ — (b)

(a) + (b)

$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = \sec^2 u\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)$

$$\tan u = \sec^2 u \left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)$$

$$\text{or, } \frac{\tan u}{\sec^2 u} = x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$$

$$\text{or, } \frac{\frac{\sin u}{\cos u}}{\frac{1}{\cos^2 u}} = x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$$

$$\text{or, } \frac{\sin u}{\cancel{\cos u}} \times \cos^2 u = x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$$

$$\text{or, } \frac{2\sin u \times \cos u}{2} = \cancel{\frac{1}{2}\sin 2u} \; x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$$

$$\text{or, } \frac{1}{2}\sin 2u = x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}.$$

(32) Q.No → If $u = \sec^{-1}\frac{x^2+y^2}{x-y}$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \cot u$

Ans. → $\because u = \sec^{-1}\frac{x^2+y^2}{x-y}$

$$\therefore \sec u = \frac{x^2+y^2}{x-y} = \frac{x^2\left(1+\frac{y^2}{x^2}\right)}{x\left(1-\frac{y}{x}\right)} = x f\frac{y}{x} = v \text{ -①}$$

Hence v is a homogeneous function of x and y of degree one.

Hence, from Euler's theorem.

$$x\frac{\partial v}{\partial x}+y\frac{\partial v}{\partial y}=1\times v$$

Now, $v=\sec u$ — ②

∴ Diff. ② partially w.r.t. x, we have,

$$\frac{\partial v}{\partial x}=\sec u.\tan u\frac{\partial u}{\partial x}$$

$$x\frac{\partial v}{\partial x}=x\sec u.\tan u\frac{\partial u}{\partial x}\quad \text{— ⓐ}$$

Similarly $y\frac{\partial v}{\partial y}=y\sec u.\tan u\frac{\partial u}{\partial y}$ — ⓑ

ⓐ + ⓑ

$$x\frac{\partial v}{\partial x}+y\frac{\partial v}{\partial y}=\sec u.\tan u\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)$$

$$v=\sec u=\sec u.\tan u\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)$$

or, $\frac{\sec u}{\sec u.\tan u}=\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)$

or, $\frac{\frac{1}{\cos u}}{\frac{1}{\cos u}.\frac{\sin u}{\cos u}}=\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)$

or, $\frac{\cancel{1}}{\cancel{\cos u}}\times\frac{\cancel{\cos u}}{\cancel{1}}.\frac{\cos u}{\sin u}=\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)$

$\cot u=x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}$ Proved.

33 QN.→ If $v=\tan^{-1}\frac{x^3+y^3}{x-y}$, then show that

$$x\frac{\partial v}{\partial x}+y\frac{\partial v}{\partial y}=\sin 2v.$$

Ans.→ $\because\ v = \tan^{-1}\dfrac{x^3+y^3}{x-y}$

$$\therefore \tan v = \frac{x^3+y^3}{x-y} = \frac{x^3\left(1+\frac{y^3}{x^3}\right)}{x\left(1-\frac{y}{x}\right)} = x^2 f\left(\frac{y}{x}\right) = u \quad (1)$$

Hence u is a homogeneous function of x and y of degree two.

Hence, from Euler's theorem

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu = 2u -$$

Now, $u = \tan v$ — (2)

$\therefore$ Diff. (2) partially w.r.t. x we have,

$$\frac{\partial u}{\partial x} = \sec^2 v \frac{\partial v}{\partial x}$$

$$x\frac{\partial u}{\partial x} = x\sec^2 v\frac{\partial v}{\partial x} \quad \text{— (a)}$$

Similarly $y\dfrac{\partial u}{\partial y} = y\sec^2 v\dfrac{\partial v}{\partial y}$ — (b)

(a) + (b)

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sec^2 v\left(x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y}\right)$$

$$2u = 2\tan v = \sec^2 v\left(x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y}\right)$$

or, $\dfrac{2\tan v}{\sec^2 v} = x\dfrac{\partial v}{\partial x} + y\dfrac{\partial v}{\partial y}$

or, $\dfrac{2\frac{\sin v}{\cos v}}{\frac{1}{\cos^2 v}} = x\dfrac{\partial v}{\partial x} + y\dfrac{\partial v}{\partial y}$

$$\text{or, } 2\frac{\sin v}{\cos v} \times \frac{\cos^2 v}{1} = x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y}$$

$$\text{or, } 2\sin v \cdot \cos v = x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y}$$

$$\therefore \sin 2v = x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} \text{ Proved.}$$